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ON THICKNESS SHEAR DEFORMATION THEORIES FOR THE DYNAMIC ANALYSIS OF NON-CIRCULAR CYLINDRICAL SHELLS

KOSTAS P. SOLDATOS
Department of Applied Mathematics, University of Ioannina, Ioannina 45332, Greece

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Abstract—This paper is concerned with the problem of free vibrations of homogeneous isotropic non-circular cylindrical shells, including the effects of thickness shear deformation and rotatory inertia. For this problem the equations of motion of two first approximation shell theories are derived. Both theories are transverse shear deformable analogues of the classical Love-type theory. The first theory involves thickness shear correction factors while the second one assumes a parabolic variation for thickness shear strains and stresses, with zero values at the inner and outer shell surfaces. The equations of both theories are solved, for the case of a simply supported non-circular cylindrical shell and, as an application, the free vibration problem of a simply supported oval cylindrical shell is considered. From comparisons made between corresponding numerical results based on both theories, as well as the classical Love-type theory, a superiority of the theory assuming parabolic variation of thickness shear is concluded.

1. INTRODUCTION

The free vibration problem of non-circular cylindrical shells has been studied by several investigators[1–10]. In all these articles, classical thin shell theories based on the Kirchhoff–Love assumptions have been used. Donnell-type quasi-shallow shell theory[11] has been used by Culberson and Boyd[2], Soldatos and Tzivanidis[5] and Soldatos[8]; Love-type first approximation theory[12] has been used by Culberson and Boyd[2] and Shirakawa and Morita[6]; Sanders-type best first approximation theory has been used by Sewall and Pusey[1], Elsbernd and Leissa[3] and Chen and Kempner[4]; the Goldenveizer–Novozhilov approximations[13, 14] have been employed by Yamada *et al.*[10]; finally, Flugge-type second approximation shell theory[15] has been used by Koumousis and Armenakas[7] and Soldatos[9].

The inclusion of thickness shear deformation effects into a theory suitable for the vibration analysis of non-circular cylindrical shells is essential. Especially, it is more essential in cases of laminated composite materials[5, 8, 9] in which the ratio of the thickness shear moduli to their in-plane Young's moduli can be much smaller than the corresponding ratio in homogeneous isotropic materials.

In order to include thickness shear deformation effects into the corresponding problem of homogeneous circular cylindrical shells, Herrmann and Mirsky[16, 17] derived second approximation equations and Warburton and Soni[18] derived first approximation equations for the isotropic and orthotropic case, respectively. For laminated composite cases several sets of equations have been derived; by Sinha and Rath[19] which used Donnell's approximations, by Dong and Tso[20] and Rath and Das[21] which used Love's approximations, and by Hsu *et al.*[22] which used Sanders' approximations (see also Refs [23, 24]).

The derivation of all sets of equations presented in Refs [16–22] was guided by the work of Mindlin[25] in the theory of homogeneous isotropic plates and, therefore, led to the introduction of shear correction factors in the transverse shear resultant–strain relation. In Ref. [25] the value of these correction factors had been selected as $\pi^2/12$. In the case of cylindrical shells (either homogeneous or laminated composites) these factors account not only for the variations of shear angles and complex stress state but also for the types of materials, the manner in which they are assembled as well as the geometric characteristics of the particular shell element. Hence, a procedure of determining these factors never gives absolutely acceptable results while it is always cumbersome. In order to avoid this difficulty Bhimaraddi[26], using second-order approximations, and Reddy and Liu[27], using first-order approximations, have recently derived the equations of some theories which take into consideration thickness shear deformation effects while do not involve the determination of any unknown shear correction factors. This is achieved on expanding the shell middle surface displacement components, in power series of the transverse coordinate, in a manner which permits a realistic parabolic variation for thickness shear strains and stresses, with zero values at the extreme fibres.

This paper can be considered as one of the primary attempts in the field of shell theories which, taking into account thickness shear deformation effects, are suitable for the free vibration analysis of non-circular cylindrical shells. As such, it is concerned with the homogeneous isotropic case only. Two first approximation shell theories are proposed and their equations of motion are derived. Both theories are transverse shear deformable analogues of the classical Love-type theory. The first theory involves thickness shear correction factors; in the special case of a circular cylinder, its equations reduce to the homogeneous isotropic version of Warburton and Soni's[18] and Dong and Tso's[20] equations. The second theory is motivated by the assumptions made in Refs [26, 27]. It assumes a parabolic variation of thickness shear and, therefore, does not involve any shear correction factors which, in the non-circular cylindrical case, must also account for a non-zero eccentricity parameter characterizing the cylindrical cross-section.

As an application, the equations of motion of both theories are solved for simply supported non-circular shells. The solution, based on the application of Galerkin's method, is obtained in a generalized algebraic eigenvalue form and is independent of the profile of the cylindrical cross-section. As a special case of the obtained solution, the free vibration problem of simply supported oval shells is considered. For this problem, numerical results based on both theories, as well as the Love-type classical theory[2], are obtained, compared and discussed.

2. THEORETICAL FORMULATION

The nomenclature of the middle surface of a moderately thick non-circular cylindrical shell is shown in Fig. 1. According to the general procedure outlined in Refs [5, 9], a function, $f(s)$, of the circumferential coordinate is considered to describe the divergence of the non-circular shell configuration from that of a corresponding circular shell. Denoting by R_0 the constant radius of the circular shell, the curvature of the non-circular shell is given as

$$(1/R) = (1/R_0)f(s) \tag{1}$$

where R is its variable radius of curvature.

The shell is composed by a homogeneous isotropic linearly elastic material whose state of stress is governed by Hooke's law

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_s), \qquad \sigma_s = \frac{E}{1-\nu^2}(\varepsilon_s + \nu\varepsilon_x),$$
$$(\tau_{xs}, \tau_{xz}, \tau_{sz}) = \frac{E}{2(1+\nu)}(\varepsilon_{xs}, \varepsilon_{xz}, \varepsilon_{sz}) \tag{2}$$

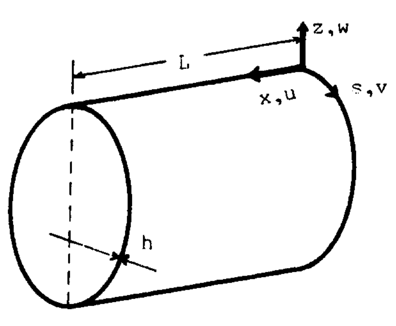

Fig. 1. Nomenclature of a non-circular cylindrical shell.

where E and ν denote Young's modulus and Poisson's ratio, respectively. Disregarding in eqns (2) the contribution of transverse shear strains ($\varepsilon_{xz} = \varepsilon_{sz} = 0$), the two-dimensional Hooke's law which governs the plane stress state in a homogeneous isotropic linearly elastic material is obtained.

2.1. *A theory employing thickness shear correction factors*

In order to take into consideration the effect of transverse shear deformation, consider the following displacement field

$$
\begin{aligned}
U(x,s,z;t) &= u(x,s;t)+z\psi_x(x,s;t),\\
V(x,s,z;t) &= v(x,s;t)+z\psi_s(x,s;t),\\
W(x,s,z;t) &= w(x,s;t)
\end{aligned}
\tag{3}
$$

where t denotes time. According to Timoshenko's beam[28, 29] and Mindlin's plate[25] theories, u, v and w are the shell middle surface displacement components, while ψ_x and ψ_s represent the angular rotation, about the s and x directions, respectively, of straight lines normal to the middle surface before deformation. These lines remain straight but, due to the transverse shear consideration, they do not remain normal to the middle surface after deformation.

Using the displacement field, eqns (3), and for Love's first approximation shell theory used here[12, 30], the strains appearing in eqns (2) can be decomposed into extensional and flexural components according to

$$
\varepsilon_x = e_x+zk_x, \qquad \varepsilon_s = e_s+zk_s, \qquad \varepsilon_{xs} = e_{xs}+zk_{xs}, \qquad \varepsilon_{xz} = e_{xz}, \qquad \varepsilon_{sz} = e_{sz} \tag{4}
$$

where

$$
\begin{aligned}
&e_x = u_{,x}, \qquad e_s = v_{,s}+w/R, \qquad e_{xs} = v_{,x}+u_{,s} \qquad e_{xz} = \psi_x+w_{,x},\\
&e_{sz} = \psi_s+w_{,s}-\frac{v}{R}, \qquad k_x = \psi_{x,x}, \qquad k_s = \psi_{s,s}, \qquad k_{xs} = \psi_{s,x}+\psi_{x,s}.
\end{aligned}
\tag{5}
$$

For thin shells obeying Kirchhoff–Love assumptions, the requirement of negligible shear strains ($e_{xz} = e_{sz} = 0$) gives

$$
\psi_x = -w_{,x}, \qquad \psi_s = -w_{,s}+\frac{v}{R} \tag{6}
$$

and the ensuing consequence of eqns (5) result in forms appropriate to the classical Love-type theory[2, 6, 31].

The force and moment resultants are, respectively, defined as

$$
\begin{aligned}
(N_x, N_s, N_{xs}, Q_x, Q_s) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs}, \tau_{xz}, \tau_{sz})\, \mathrm{d}z,\\
(M_x, M_s, M_{xs}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs}) z\, \mathrm{d}z
\end{aligned}
\tag{7}
$$

where h is the shell constant thickness. After introduction of eqns (2) into eqns (7) and carrying out the denoted integrations, the following constitutive equations are obtained

$$
\begin{aligned}
&N_x = C(e_x+\nu e_s), \qquad N_s = C(e_s+\nu e_x), \qquad (N_{xs}, Q_x, Q_s) = C\frac{1-\nu}{2}(e_{xs}, k_5 e_{xz}, k_4 e_{sz}),\\
&M_x = D(k_x+\nu k_s), \qquad M_s = D(k_s+\nu k_x), \qquad M_{xs} = D\frac{1-\nu}{2}k_{xs}
\end{aligned}
\tag{8}
$$

where k_4 and k_5 are thickness shear correction constants and the extensional and bending rigidities of the shell are given according to

$$C = Eh/(1-\nu^2), \qquad D = Eh^3/12(1-\nu^2). \tag{9}$$

For the thickness shear deformable analogue of Love's first approximation shell theory, the equations of motion, governing the free vibration problem of cylindrical shells are given in Ref. [20]. In the present notation, they can be expressed as

$$\begin{aligned} N_{x,x}+N_{xs,s} &= \rho_0 u_{,tt}, \\ N_{xs,x}+N_{s,s}+Q_s/R &= \rho_0 v_{,tt}, \\ Q_{x,x}+Q_{s,s}-N_s/R &= \rho_0 w_{,tt}, \\ M_{x,x}+M_{xs,s}-Q_x &= \rho_2 \psi_{x,tt}, \\ M_{xs,x}+M_{s,s}-Q_s &= \rho_2 \psi_{s,tt} \end{aligned} \tag{10}$$

where

$$\rho_i = \int_{-h/2}^{h/2} \rho z^i \, \mathrm{d}z \tag{11}$$

and ρ is the constant mass density of the shell material.

After eqns (5) and (8), eqns (10) can be expressed in the following differential eigenvalue form

$$[\mathscr{L}]\{\delta\} = 0 \tag{12}$$

where

$$\{\delta\}^{\mathrm{T}} = \{u, v, w, R_0\psi_x, R_0\psi_s\} \tag{13}$$

and $[\mathscr{L}]$ is a 5×5 matrix of linear partial differential operators; a non-dimensional version of its components is given in Appendix A, in terms of the following parameters

$$\begin{aligned} &\eta = x/L, \quad \xi = s/2\pi R_0 \quad (0 \leqslant \eta, \xi \leqslant 1), \quad \lambda = R_0/L, \quad \nu_1 = (1-\nu)/2, \\ &\nu_2 = (1+\nu)/2, \quad \bar{D} = h^2/12R_0^2, \quad \bar{\rho} = \rho_0(1-\nu^2)R_0^2/E. \end{aligned} \tag{14}$$

The constants k_4 and k_5, appearing in eqns (8), are the analogues of the well-known thickness shear correction factors introduced by Mindlin[25] for the analysis of homogeneous isotropic plates; their value had been estimated as $\pi^2/12$. This value has also been found suitable for the analysis of homogeneous circular cylindrical shells[17, 18, 32, 33]. The same value ($\pi^2/12$) had further been used, for conveniency, in some analyses of laminated composite circular cylindrical shells[34, 35] where higher order shell theories had been employed, regardless that, in those analyses, the correction factors must account not only for the variations of shear angles and complex stress state but also for the types of materials and the manner in which they are assembled.

In the present study, the correction factors appearing in eqns (8) must also account for the eccentricity characterizing the shape of the cylindrical cross-section. Hence, a procedure for determining these factors seems to be very cumbersome while is, in general, dependent on the particular shape of the cross-section of the shell considered.

In the example of the oval cylindrical shell considered in Section 4, in order to avoid the difficulty of the determination of the shear correction factors, an attempt was made to use their Mindlin's value ($k_4 = k_5 = \pi^2/12$). However, some of the obtained numerical results showed that this choice of their value is insufficient for a detailed study of the free vibration problem of moderately thick non-circular cylindrical shells. This makes apparent the necessity of the shell theory presented in the next section, which employs parabolic variation of thickness shear and, therefore, does not involve any thickness shear correction factors.

2.2. *A theory employing parabolic variation for thickness shear*

Instead of the displacement field, eqns (3), the following displacement expansion is now employed

$$\begin{aligned} U(x,s,z;t) &= u(x,s;t)-z[w_{,x}+\zeta u_1(x,s;t)], \\ V(x,s,z;t) &= (1+z/R)v(x,s;t)-z[w_{,s}+\zeta v_1(x,s;t)], \\ W(x,s,z;t) &= w(x,s;t) \end{aligned} \tag{15}$$

where $\zeta = (4z^2/3h^2-1)$. Again u, v and w represent the shell middle surface displacement components. The terms $zw_{,x}$ and $z(w_{,s}-v/R)$ are the standard terms which guarantee the validity of the Kirchhoff–Love assumptions in the classical Love-type thin shell theory. The remaining terms, including those involving the unknown functions u_1 and v_1, have been employed to disturb the assumption that normals to the undeformed middle surface still remain normal to it after deformation; they also remove the assumption, introduced by the displacement fields, eqns (3), that these normals remain straight after deformation. Instead, as it can be seen from eqns (17) below, they lead to a realistic parabolic variation for thickness shear strains and stresses, with zero values at both the inner and outer shell surfaces.

The displacement field, eqns (15), is introduced into the linear version of the strain–displacement relations of the three-dimensional elasticity[30, 36]

$$\begin{gathered} \varepsilon_x = U_{,x}, \quad \varepsilon_s = (1+z/R)^{-1}(V_{,s}+W/R), \quad \varepsilon_z = W_{,z}, \\ \varepsilon_{xs} = V_{,x}+(1+z/R)^{-1}U_{,s}, \quad \varepsilon_{xz} = W_{,x}+U_{,z}, \\ \varepsilon_{sz} = (1+z/R)^{-1}(W_{,s}-V/R)+V_{,z}. \end{gathered} \tag{16}$$

Restricting next our approximations in the limits of a first-order theory ($h/R \ll 1$), the strains appearing in eqns (2) can be expressed, in a power series of the transverse coordinate z, as

$$\begin{gathered} \varepsilon_x = e_x+zk_x-\frac{4z^3}{3h^2}m_x, \quad \varepsilon_{xs} = e_{xs}+zk_{xs}-\frac{4z^3}{3h^2}m_{xs}, \quad \varepsilon_s = e_s+zk_s-\frac{4z^3}{3h^2}m_s, \\ \varepsilon_{xz} = \left(1-\frac{4z^2}{h^2}\right)e_{xz}, \quad \varepsilon_{sz} = \left(1-\frac{4z^2}{h^2}\right)e_{sz} \end{gathered} \tag{17}$$

where

$$\begin{gathered} e_x = u_{,x}, \quad e_s = v_{,s}+w/R, \quad e_{xs} = v_{,x}+u_{,s}, \quad e_{xz} = u_1, \quad e_{sz} = v_1, \\ k_x = u_{1,x}-w_{,xx}, \quad k_s = v_{1,s}-w_{,ss}+(v/R)_{,s}, \quad k_{xs} = v_{1,x}+u_{1,s}-2w_{,xs}+v_{,x}/R, \\ m_x = u_{1,x}, \quad m_s = v_{1,s}, \quad m_{xs} = v_{1,x}+u_{1,s}. \end{gathered} \tag{18}$$

As it now becomes apparent the unknown functions u_1 and v_1 represent the thickness shear strains action on the shell middle surface. For thin shells obeying Kirchhoff–Love assumptions, the requirement of negligible shear strains ($e_{xz} = e_{sz} = 0$) leads to $u_1 = v_1 = 0$ and the ensuing consequence of eqns (18) result in forms appropriate to the classical Love-type theory[2, 6, 31].

Force and moment resultants are defined as

$$
\begin{aligned}
(N_x, N_s, N_{xs}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs})\,\mathrm{d}z,\\
(Q_x, Q_s) &= \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{sz})(1-4z^2/h^2)\,\mathrm{d}z,\\
(M_x, M_s, M_{xs}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs})z\,\mathrm{d}z,\\
(S_x, S_s, S_{xs}) &= \frac{4}{3h^2}\int_{-h/2}^{h/2} (\sigma_x, \sigma_s, \tau_{xs})z^3\,\mathrm{d}z.
\end{aligned}
\tag{19}
$$

Introduction of eqns (2) into eqns (19) leads to the following constitutive equations

$$
\begin{aligned}
&N_x = C(e_x + ve_s), \qquad N_s = C(e_s + ve_x),\\
&(N_{xs}, Q_x, Q_s) = C\frac{1-v}{2}\left(e_{xs}, \frac{8}{15}e_{xz}, \frac{8}{15}e_{sz}\right),\\
&M_x = D[k_x + vk_s - (m_x + vm_s)/5],\\
&S_x = D[(k_x + vk_s)/5 - (m_x + vm_s)/21],\\
&M_s = D[k_s + vk_x - (m_s + vm_x)/5],\\
&S_s = D[(k_s + vk_x)/5 - (m_s + vm_x)/21],\\
&M_{xs} = D\frac{1-v}{2}(k_{xs} - m_{xs}/5),\\
&S_{xs} = D\frac{1-v}{2}(k_{xs}/5 - m_{xs}/21).
\end{aligned}
\tag{20}
$$

The equations of motion are derived by employing Hamilton's principle and using the standard procedure of the calculus of variations. Accordingly, the following differential equations, governing the free vibration problem, are obtained

$$
\begin{aligned}
&N_{x,x} + N_{xs,s} = \rho_0 u_{,tt},\\
&N_{s,s} + N_{xs,x} + (M_{s,s} + M_{xs,x})/R = \rho_0 v_{,tt} + \frac{\rho_2}{R}(v_1 - w_{,s} + v/R)_{,tt} - \frac{4\rho_4}{3h^2R}v_{1,tt},\\
&-N_s/R + M_{x,xx} + M_{s,ss} + 2M_{xs,xs} = \rho_0 w_{,tt} - \rho_2[w_{,xx} + w_{,ss} - 4(u_{1,x} + v_{1,s})/5]_{,tt} - \frac{4\rho_4}{3h^2}(u_{1,x} + v_{1,s})_{,tt},\\
&(M_x - S_x)_{,x} + (M_{xs} - S_{xs})_{,s} - Q_x = \frac{4}{5}\rho_2\left(\frac{17}{21}u_1 - w_{,x}\right)_{,tt} + \frac{4\rho_4}{3h^2}(w_{,x} - 2u_1)_{,tt} + \frac{16}{9h^4}\rho_6 u_{1,tt},\\
&(M_s - S_s)_{,s} + (M_{xs} - S_{xs})_{,x} - Q_s = \frac{4}{5}\rho_2\left(\frac{17}{21}v_1 - w_{,s}\right)_{,tt} + \frac{4\rho_4}{3h^2}(w_{,s} - v/R - 2v_1)_{,tt} + \frac{16}{9h^4}\rho_6 v_{1,tt}
\end{aligned}
\tag{21}
$$

associated by the following boundary conditions, which must be satisfied on the shell boundaries $\eta = 0, 1$:

$$\begin{aligned}
&u \text{ prescribed or } N_x = 0,\\
&v \text{ prescribed or } N_{xs} + M_{xs}/R = 0,\\
&w \text{ prescribed or } M_{x,x} + 2M_{xs,s} = 0,\\
&w_{,x} \text{ prescribed or } M_x = 0,\\
&u_1 \text{ prescribed or } M_x - S_x = 0,\\
&v_1 \text{ prescribed or } M_{xs} - S_{xs} = 0.
\end{aligned} \tag{22}$$

After eqns (18) and (20), eqns (21) can also be expressed in the differential eigenvalue form (12), where now it is

$$\{\delta\}^{\mathrm{T}} = \{u, v, w, R_0 u_1, R_0 v_1\}. \tag{23}$$

A non-dimensional version of the components of the linear differential operator matrix $[\mathscr{L}]$ is given in Appendix A, in terms of the parameters defined in eqns (14).

3. SOLUTION FOR SIMPLY SUPPORTED SHELLS

The case of a shell subjected to the following set of edge boundary conditions, at $\eta = 0$, 1, is considered:

$$N_x = M_x = v = w = \psi_s = 0 \tag{24}$$

for the equations of the theory proposed in Section 2.1, or

$$N_x = M_x = S_x = v = w = v_1 = 0 \tag{25}$$

for the equations of the theory proposed in Section 2.2.

Both of the aforementioned sets of boundary conditions are equivalent to the set of S2 simply supported edge boundary conditions considered in Refs [5, 9] and elsewhere. They are exactly satisfied on assuming the following displacement model

$$\begin{aligned}
u &= \cos(\omega t)\cos(m\pi\eta)\sum_{n=\{^1_0\}}^{N} a_n \begin{Bmatrix}\sin(2n\pi\xi)\\ \alpha\cos(2n\pi\xi)\end{Bmatrix},\\
v &= \cos(\omega t)\sin(m\pi\eta)\sum_{n=\{^0_1\}}^{N} b_n \begin{Bmatrix}\alpha\cos(2n\pi\xi)\\ \sin(2n\pi\xi)\end{Bmatrix},\\
w &= \cos(\omega t)\sin(m\pi\eta)\sum_{n=\{^1_0\}}^{N} c_n \begin{Bmatrix}\sin(2n\pi\xi)\\ \alpha\cos(2n\pi\xi)\end{Bmatrix},\\
R_0(\psi_x, u_1) &= \cos(\omega t)\cos(m\pi\eta)\sum_{n=\{^1_0\}}^{N} (d_n, D_n) \begin{Bmatrix}\sin(2n\pi\xi)\\ \alpha\cos(2n\pi\xi)\end{Bmatrix},\\
R_0(\psi_s, v_1) &= \cos(\omega t)\sin(m\pi\eta)\sum_{n=\{^0_1\}}^{N} (e_n, E_n) \begin{Bmatrix}\alpha\cos(2n\pi\xi)\\ \sin(2n\pi\xi)\end{Bmatrix}.
\end{aligned} \tag{26}$$

Here, ω represents a certain unknown natural frequency, m and $2n$ are the axial and circumferential half-wave numbers, respectively, a_n, b_n, c_n, d_n, D_n, e_n and E_n are unknown constant coefficients and α is equal to 1/2 if $n = 0$ and is equal to 1 if $n > 0$. The Fourierseries expansions (26) have similar properties with those of the displacement models used in Refs [5, 9], for a corresponding classical shell theory problem; the upper and lower functions appearing in the braces represent antisymmetric and symmetric displacements, respectively, in the shell circumferential direction.

Introduction of the displacement model, eqns (26), into the differential equations (12) and application, in ξ direction, of the method of Galerkin (with $i = 0, 1, 2, \ldots, N$),

$$\begin{aligned}
&\int_0^1 [\mathscr{L}_{11}(u)+\mathscr{L}_{12}(v)+\mathscr{L}_{13}(w)+R_0\mathscr{L}_{14}(\psi_x,u_1)+R_0\mathscr{L}_{15}(\psi_s,v_1)]\begin{Bmatrix}\sin(2i\pi\xi)\\ \cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi = 0,\\
&\int_0^1 [\mathscr{L}_{21}(u)+\mathscr{L}_{22}(v)+\mathscr{L}_{23}(w)+R_0\mathscr{L}_{24}(\psi_x,u_1)+R_0\mathscr{L}_{25}(\psi_s,v_1)]\begin{Bmatrix}\cos(2i\pi\xi)\\ \sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi = 0,\\
&\int_0^1 [\mathscr{L}_{31}(u)+\mathscr{L}_{32}(v)+\mathscr{L}_{33}(w)+R_0\mathscr{L}_{34}(\psi_x,u_1)+R_0\mathscr{L}_{35}(\psi_s,v_1)]\begin{Bmatrix}\sin(2i\pi\xi)\\ \cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi = 0, \qquad (27)\\
&\int_0^1 [\mathscr{L}_{41}(u)+\mathscr{L}_{42}(v)+\mathscr{L}_{43}(w)+R_0\mathscr{L}_{44}(\psi_x,u_1)+R_0\mathscr{L}_{45}(\psi_s,v_1)]\begin{Bmatrix}\sin(2i\pi\xi)\\ \cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi = 0,\\
&\int_0^1 [\mathscr{L}_{51}(u)+\mathscr{L}_{52}(v)+\mathscr{L}_{53}(w)+R_0\mathscr{L}_{54}(\psi_x,u_1)+R_0\mathscr{L}_{55}(\psi_s,v_1)]\begin{Bmatrix}\cos(2i\pi\xi)\\ \sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi = 0
\end{aligned}$$

leads to a general eigenvalue problem of the form

$$\left(\begin{bmatrix}\mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} & \mathbf{T}_{14} & \mathbf{T}_{15}\\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} & \mathbf{T}_{24} & \mathbf{T}_{25}\\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} & \mathbf{T}_{34} & \mathbf{T}_{35}\\ \mathbf{T}_{41} & \mathbf{T}_{42} & \mathbf{T}_{43} & \mathbf{T}_{44} & \mathbf{T}_{45}\\ \mathbf{T}_{51} & \mathbf{T}_{52} & \mathbf{T}_{53} & \mathbf{T}_{54} & \mathbf{T}_{55}\end{bmatrix} - \bar{\omega}^2 \begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{22} & \mathbf{H}_{23} & \mathbf{0} & \mathbf{H}_{25}\\ \mathbf{0} & \mathbf{H}_{32} & \mathbf{H}_{33} & \mathbf{H}_{34} & \mathbf{H}_{35}\\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{43} & \mathbf{H}_{44} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}_{52} & \mathbf{H}_{53} & \mathbf{0} & \mathbf{H}_{55}\end{bmatrix}\right)\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\\ \mathbf{C}\\ \mathbf{D}\\ \mathbf{E}\end{bmatrix} = \{0\} \qquad (28)$$

where $\bar{\omega}^2 = \bar{\rho}\omega^2$ is a squared non-dimensional frequency parameter and **I** and **0** represent proper unit and zero matrices, respectively. The components of the square matrices $\mathbf{T}_{ij}$, $\mathbf{H}_{ij}$ and the column matrices **A**, **B**, **C**, **D** and **E** are given in Appendix B.

The eigenvalue problem, eqn (28), solved by a standard numerical procedure, gives $5N+2$ or $5N+3$ eigenvalues for antisymmetric of symmetric displacements, respectively. Each one of these eigenvalues is an approximation for a corresponding squared non-dimensional frequency of the shell. Thus, the integer N must be chosen so that, for the obtained numerical results, convergence be ensured to a desired accuracy.

From the obtained $5N+2$ or $5N+3$ frequencies, five are associated with each mode whose nominal circumferential half-wave number n is 1, 2, . . . or N; two of them are in-plane vibration frequencies (predominant influence in the axial and circumferential shell directions x and s), one of them is a flexural vibration frequency (predominant influence normal to the shell middle surface direction, z), while the other two are thickness shear vibration frequencies (shear predominant influence in x–z and s–z planes). With $n = 0$ and antisymmetric displacements (torsional modes) two natural frequencies are associated; an in-plane frequency with predominant influence in the circumferential direction and a thickness shear one with predominant influence in the distortion of the s–z plane. With $n = 0$ and symmetric displacements (breathing modes) three natural frequencies are associated; an in-plane, a flexural and a thickness-shear one, with predominant influence in the axial direction x, the normal to the middle surface direction z and shear deformation in the x–z plane, respectively.

4. FREE VIBRATIONS OF SHELLS OF OVAL CROSS-SECTION

In order to test both theories proposed in Section 2, as well as the reliability of corresponding numerical results obtained on their basis, the aforementioned analysis is applied in a particular case, in which the simply supported non-circular cylindrical shell has an oval cross-section. According to the oval curvature representation introduced by Romano and Kempner[37, 38], the function $f(\xi)$ takes the form

$$f(\xi) = 1 + \varepsilon \cos(4\pi\xi) \tag{29}$$

where ε is an eccentricity parameter such that $|\varepsilon| \leqslant 1$.

Expression (29) represents a doubly symmetric oval configuration. As a result, all frequencies obtained from the numerical solution of the eigenvalue problem (28) can be classified into four groups, each one of which is associated with one of four types of uncoupled natural modes of vibration; this is explained, in detail, in Refs [2, 3, 7, 9] (see also Appendix B). Each one of these groups of frequencies occur for one of the following displacement models:

(1) antisymmetric displacements with even circumferential half-wave numbers ($n = 0$, 2, 4, . . .),

(2) antisymmetric displacements with odd circumferential half-wave numbers ($n = 1$, 3, 5, . . .),

(3) symmetric displacements with even circumferential half-wave numbers ($n = 0$, 2, 4, . . .),

(4) symmetric displacements with odd circumferential half-wave numbers ($n = 1$, 3, 5, . . .).

Furthermore, the whole frequency spectra obtained for any two opposite values of the eccentricity parameter ($\pm\varepsilon$) are identical. Accordingly, the range of variation of ε is limited to $0 \leqslant \varepsilon \leqslant 1$.

4.1. *Numerical results and discussion*

All numerical results presented throughout this study are for oval shells with Poisson's ratio $\nu = 0.25$. The indication '$k_4 = k_5 = \pi^2/12$' or 'parabolic shear' denotes numerical results obtained on the basis of the theory using thickness shear correction factors or the theory assuming parabolic variation of thickness shear, respectively. Numerical results denoted with the indication 'classical theory' were obtained on the basis of the first approximation Love-type equations presented in Ref. [2].

Corresponding non-dimensional flexural and in-plane torsional vibration frequencies, $\bar{\omega}$, obtained on the basis of all three of the afore-mentioned theories, are cited in Tables 1 and 2, for a shell with $L/mR_0 = 6$ and several values of h/R_0. A relatively small value of the eccentricity parameter ($\varepsilon = 0.2$) has been considered for the results shown in Table 1, while a large value of it ($\varepsilon = 1$) has been considered for the results shown in Table 2. The in-plane torsional vibration frequencies, obtained by using antisymmetric displacements ($n = 0$) are enclosed in parentheses.

Comparisons between corresponding numerical results show that the parabolic shear theory predicts lower frequencies than the classical theory does. This is in accordance with the generally known observation that, due to the neglect of thickness shear effects, classical plate and shell theories overpredict natural frequencies of vibration. However, the observed discrepancies, between classical and parabolic shear theories, do not exceed the engineering admissible error (5%) even for $h/R_0 \geqslant 0.05$ (this value of thickness to radius ratio is usually considered as an upper limit for the validity of classical shell theories concerned with homogeneous isotropic shells[14]).

On the other hand, and independently of the value of h/R_0, remarkable discrepancies between corresponding results based on the classical theory and the theory using shear correction factors occur for antisymmetric displacements only, with $n = 2$ and 0 (in-plane torsional frequency). For instance, for $\varepsilon = 0.2$ and $n = 2$ this discrepancy is about 8, 5 and

Table 1. Non-dimensional flexural and in-plane torsional vibration frequencies $\bar{\omega}$ ($\nu = 0.25$, $L/mR_0 = 6$, $\varepsilon = 0.2$)

		Antisymmetric displacements			Symmetric displacements		
h/R_0	n	Classical Theory	$k_4=k_5=\frac{\pi^2}{12}$	Parabolic shear	Classical Theory	$k_4=k_5=\frac{\pi^2}{12}$	Parabolic shear
0.025	0	(0.3207)	(0.3254)	(0.3207)	0.5016	0.5016	0.5016
	1	0.1364	0.1364	0.1363	0.1496	0.1496	0.1496
	2	0.0590	0.0641	0.0589	0.1060	0.1060	0.1059
	3	0.0623	0.0622	0.0622	0.0624	0.0623	0.0623
	4	0.1078	0.1077	0.1076	0.1083	0.1082	0.1081
	5	0.1718	0.1713	0.1713	0.1718	0.1713	0.1713
	6	0.2509	0.2499	0.2499	0.2509	0.2499	0.2499
0.050	0	(0.3207)	(0.3254)	(0.3207)	0.5016	0.5016	0.5016
	1	0.1366	0.1366	0.1366	0.1497	0.1497	0.1496
	2	0.0692	0.0735	0.0691	0.1124	0.1123	0.1123
	3	0.1154	0.1150	0.1148	0.1158	0.1153	0.1153
	4	0.2138	0.2123	0.2122	0.2138	0.2124	0.2123
	5	0.3430	0.3393	0.3392	0.3430	0.3393	0.3392
	6	0.5015	0.4939	0.4936	0.5015	0.4939	0.4936
0.075	0	(0.3208)	(0.3254)	(0.3208)	0.5016	0.5016	0.5016
	1	0.1362	0.1362	0.1362	0.1496	0.1496	0.1495
	2	0.0834	0.0869	0.0832	0.1217	0.1215	0.1214
	3	0.1710	0.1696	0.1695	0.1711	0.1696	0.1695
	4	0.3200	0.3152	0.3148	0.3200	0.3152	0.3148
	5	0.5141	0.5021	0.5013	0.5141	0.5021	0.5013
	6	0.7518	0.7269	0.7252	0.7518	0.7269	0.7253

5% for $h/R_0 = 0.025$, 0.05 and 0.075, respectively. Apparently, these discrepancies are tremendously magnified for $\varepsilon = 1$. Their existence is, therefore, attributed to the value employed for the shear correction factors ($k_4 = k_5 = \pi^2/12$) which seems insufficient to account for the non-zero value of the oval eccentricity parameter. The results drawn in Figs 2 and 3 support such an explanation.

In Fig. 2, the variation of both frequency parameters ($n = 0$ and 2) is shown, vs the variation of h/R_0, for a shell with a moderate eccentricity value ($\varepsilon = 0.5$). Apparently, all numerical results obtained on the basis of the classical theory and the theory assuming parabolic variation of thickness shear exhibit an acceptable behaviour. Both theories give practically identical in-plane torsional and $n = 2$ flexural frequencies.

On the contrary, corresponding numerical results obtained on the basis of the classical theory and the theory using shear correction factors fail to be in agreement even in cases of very thin shells. This can be explained as follows. In the case of a circular shell ($\varepsilon = 0$), the two torsional frequencies (the in-plane and the thickness-shear one) are related to each other through $k_4 = \pi^2/12$ (see, for instance, Ref. [17]). This value of k_4, adopted here for the case of an oval shell, has initially disturbed both torsional frequencies; non-circularity has, consequently, transferred this disturbance to the $n = 2$ flexural frequency which, as being an order of magnitude less than the in-plane torsional frequency, has been much more affected.

This becomes clear from the results obtained for a thin shell ($h/R_0 = 0.01$) as shown in Fig. 3. Both, the classical theory and the theory assuming parabolic shear variation always give identical results. On the contrary, unless the value of ε is very small (slightly non-circular shells), the theory using correction factors gives inaccurate frequencies; their

Table 2. Non-dimensional flexural and in-plane torsional vibration frequencies $\bar{\omega}$ ($\nu = 0.25$, $L/mR_0 = 6$, $\varepsilon = 1.0$)

		Antisymmetric displacements			Symmetric displacements		
h/R_0	n	Classical Theory	$k_4=k_5=\frac{\pi^2}{12}$	Parabolic shear	Classical Theory	$k_4=k_5=\frac{\pi^2}{12}$	Parabolic shear
0.025	0	(0.3208)	(0.4242)	(0.3208)	0.5067	0.5067	0.5067
	1	0.1178	0.1178	0.1177	0.1698	0.1696	0.1696
	2	0.0596	0.1506	0.0595	0.4715	0.4715	0.4714
	3	0.0439	0.0438	0.0438	0.0588	0.0588	0.0587
	4	0.1042	0.0959	0.1040	0.1037	0.1035	0.1035
	5	0.1670	0.1666	0.1665	0.1641	0.1639	0.1638
	6	0.2456	0.2448	0.2446	0.2455	0.2446	0.2445
0.050	0	(0.3213)	(0.4244)	(0.3213)	0.5068	0.5068	0.5067
	1	0.1305	0.1303	0.1302	0.1676	0.1676	0.1675
	2	0.0689	0.1409	0.0688	0.4741	0.4737	0.4735
	3	0.0775	0.0774	0.0773	0.1104	0.1102	0.1100
	4	0.2038	0.2085	0.2024	0.1985	0.1972	0.1971
	5	0.3316	0.3281	0.3279	0.3318	0.3283	0.3281
	6	0.4905	0.4832	0.4827	0.4908	0.4836	0.4832
0.075	0	(0.3223)	(0.4247)	(0.3223)	0.5068	0.5068	0.5067
	1	0.0933	0.0931	0.0931	0.1706	0.1702	0.1702
	2	0.0810	0.1502	0.0808	0.4804	0.4798	0.4796
	3	0.1619	0.1609	0.1608	0.1606	0.1597	0.1595
	4	0.3044	0.3028	0.2996	0.2925	0.2884	0.2779
	5	0.4961	0.4847	0.4839	0.4967	0.4853	0.4845
	6	0.7345	0.7109	0.7091	0.7350	0.7109	0.7091

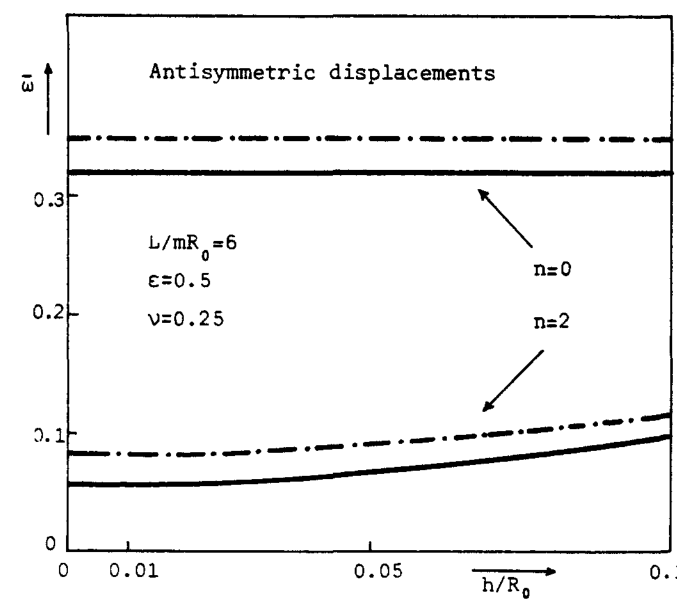

Fig. 2. Variation of the in-plane torsional ($n = 0$) and flexural ($n = 2$) frequency parameter $\bar{\omega}$ vs the ratio h/R_0 (——, classical and parabolic shear theories; —·—·—, $k_4 = k_5 = \pi^2/12$).

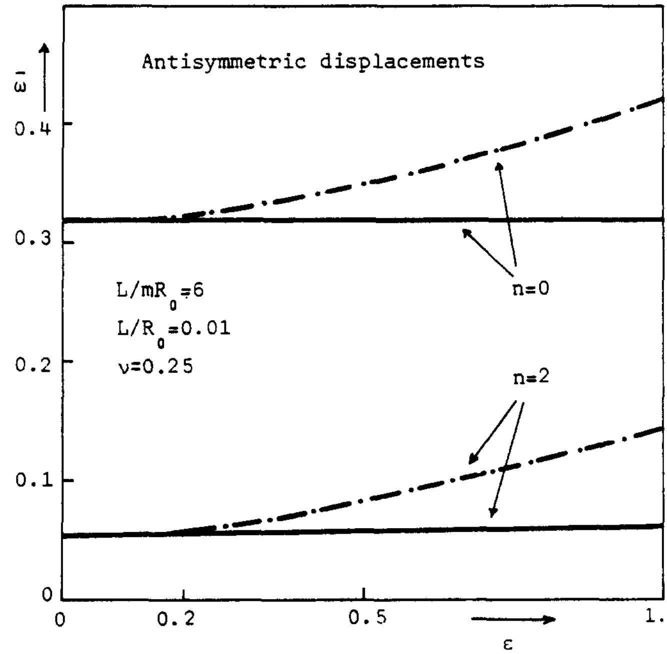

Fig. 3. Variation of the in-plane torsional ($n = 0$) and flexural ($n = 2$) frequency parameter $\bar{\omega}$ vs the eccentricity parameter ε (——, classical and parabolic shear theories; —·—·—, $k_4 = k_5 = \pi^2/12$).

inaccuracy is continuously increasing on increasing the value of the eccentricity parameter. In this aspect, the theory assuming parabolic variation of thickness shear seems to be superior and more useful than the theory using thickness shear correction factors.

5. CONCLUSIONS

The equations of motion of two first approximation shell theories, suitable for the dynamic analysis of homogeneous isotropic non-circular cylindrical shells have been derived. The first theory is an extension of Timoshenko's beam and Mindlin's plate theory and involves thickness shear correction factors. The second theory assumes a parabolic variation of thickness shear and does not make use of any shear correction factors.

The equations of both theories have been solved for simply supported non-circular shells. As an application of the obtained solution, the free vibration problem of an oval cylindrical shell has been considered. From comparisons made between corresponding numerical results based on both theories, as well as the classical Love-type theory, it has been concluded that the theory assuming parabolic variation of thickness shear seems to be superior and more useful than the theory using thickness shear correction factors.

The estimation of acceptable values for the shear correction factors is the main disadvantage of the theory which makes use of them. These factors must account not only for the variations of shear angles and complex stress state but also for the type of non-circular shell considered and the divergence of its cross-section from the configuration of the cross-section of a corresponding circular cylindrical shell.

Furthermore, in the case of a possible extension of the theory to include cross-ply laminated composite non-circular cylindrical shells, an additional difficulty in the estimation of the correction factors is that they must also account for the types of materials and the manner in which they are assembled. On the contrary, in the case of a corresponding extension of the theory assuming parabolic variation of thickness shear strain, the inner and outer shell surfaces still remain free of thickness shear stresses.

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APPENDIX A

For the thickness shear deformable analogue of the Love-type theory presented in Section 2.1, the components of the operational matrix $[\mathscr{L}]$ appearing in eqn (12) are given as

$$\begin{aligned}
\mathscr{L}_{11} &= (2\pi\lambda)^2(\ )_{,\eta\eta} + v_1(\ )_{,\xi\xi} - (2\pi)^2\bar{\rho}(\ )_{,tt},\\
\mathscr{L}_{12} &= \mathscr{L}_{21} = 2\pi\lambda v_2(\ )_{,\eta\xi},\\
\mathscr{L}_{13} &= \mathscr{L}_{31} = (2\pi)^2\lambda v f(\ )_{,\eta},\\
\mathscr{L}_{14} &= \mathscr{L}_{41} = \mathscr{L}_{15} = \mathscr{L}_{51} = \mathscr{L}_{24} = \mathscr{L}_{42} = 0,\\
\mathscr{L}_{22} &= (2\pi\lambda)^2 v_1(\ )_{,\eta\eta} + (\ )_{,\xi\xi} - (2\pi)^2 k_4 f^2(\ ) - (2\pi)^2\bar{\rho}(\ )_{,tt},\\
\mathscr{L}_{23} &= 2\pi(1+k_4)f(\ )_{,\xi} + 2\pi f'(\ ),\\
\mathscr{L}_{25} &= \mathscr{L}_{52} = 2\pi k_4 f(\ ),\\
\mathscr{L}_{32} &= 2\pi(1+k_4)f(\ )_{,\xi} + 2\pi k_4 f'(\ ),\\
\mathscr{L}_{33} &= (2\pi f)^2(\ ) - (2\pi\lambda)^2 k_5(\ )_{,\eta\eta} - k_4(\ )_{,\xi\xi} + (2\pi)^2\bar{\rho}(\ )_{,tt},\\
\mathscr{L}_{34} &= \mathscr{L}_{43} = -(2\pi)^2\lambda k_5(\ )_{,\eta},\\
\mathscr{L}_{35} &= \mathscr{L}_{53} = -2\pi k_4(\ )_{,\xi},\\
\mathscr{L}_{44} &= -(2\pi)^2 k_5(\ ) + (2\pi\lambda)^2\bar{D}(\ )_{,\eta\eta} + \bar{D}v_1(\ )_{,\xi\xi} - (2\pi)^2\bar{D}\bar{\rho}(\ )_{,tt},\\
\mathscr{L}_{45} &= \mathscr{L}_{54} = 2\pi\lambda\bar{D}v_2(\ )_{,\eta\xi},\\
\mathscr{L}_{55} &= -(2\pi)^2 k_4(\ ) + (2\pi\lambda)^2 v_1\bar{D}(\ )_{,\eta\eta} + \bar{D}(\ )_{,\xi\xi} - (2\pi)^2\bar{D}\bar{\rho}(\ )_{,tt}.
\end{aligned} \tag{A1}$$

Here, f is assumed to be expressed as a function of the non-dimensional circumferential coordinate ξ while a prime denotes ordinary differentiation with respect to ξ.

For the shell theory assuming parabolic variation of the thickness shear (Section 2.2), the components of the operational matrix $[\mathscr{L}]$, which are different from the corresponding components given by eqns (A1), are given according to

$$\begin{aligned}
\mathscr{L}_{22} &= [1+\bar{D}f^2][(2\pi\lambda)^2 v_1(\ )_{,\eta\eta} + (\ )_{,\xi\xi}] + 2\bar{D}ff'(\ )_{,\xi} + \bar{D}ff''(\ ) - (2\pi)^2\bar{\rho}(1-\bar{D}f^2)(\ )_{,tt},\\
\mathscr{L}_{23} &= 2\pi[f(\ )]_{,\xi} - 2\pi\lambda^2\bar{D}f(\ )_{,\eta\eta\xi} - \frac{1}{2\pi}\bar{D}f(\ )_{,\xi\xi\xi} + 2\pi\bar{\rho}\bar{D}f(\ )_{,\xi tt},\\
\mathscr{L}_{24} &= \frac{4}{5}(2\pi\lambda)\bar{D}v_2 f(\ )_{,\eta\xi},\\
\mathscr{L}_{25} &= \frac{4}{5}\bar{D}f[(2\pi\lambda)^2 v_1(\ )_{,\eta\eta} + (\ )_{,\xi\xi} - (2\pi)^2\bar{\rho}(\ )_{,tt}],\\
\mathscr{L}_{32} &= 2\pi f(\ )_{,\xi} - 2\pi\lambda^2\bar{D}[f(\ )_{,\eta\eta}]_{,\xi} - \frac{1}{2\pi}\bar{D}[f(\ )]_{,\xi\xi\xi} + 2\pi\bar{\rho}\bar{D}[f(\ )_{,tt}]_{,\xi},\\
\mathscr{L}_{33} &= (2\pi f)^2(\ ) + (2\pi)^2\lambda^4\bar{D}(\ )_{,\eta\eta\eta\eta} + 2\lambda^2\bar{D}(\ )_{,\eta\eta\xi\xi} + \frac{1}{(2\pi)^2}\bar{D}(\ )_{,\xi\xi\xi\xi},\\
&\quad + \bar{\rho}((2\pi)^2(\ )_{,tt} - \bar{D}[(2\pi\lambda)^2(\ )_{,\eta\eta tt} + (\ )_{,\xi\xi tt}]),\\
\mathscr{L}_{34} &= \mathscr{L}_{43} = -\frac{4}{5}\bar{D}\lambda[(2\pi\lambda)^2(\ )_{,\eta\eta\eta} + (\ )_{,\eta\xi\xi} - (2\pi)^2\bar{\rho}(\ )_{,\eta tt}],\\
\mathscr{L}_{35} &= \mathscr{L}_{53} = -\frac{4}{5}\bar{D}[2\pi\lambda^2(\ )_{,\eta\eta\xi} + \frac{1}{2\pi}(\ )_{,\xi\xi\xi} - (2\pi)^2\bar{\rho}(\ )_{,\xi tt}],\\
\mathscr{L}_{42} &= \frac{4}{5}(2\pi\lambda)\bar{D}v_2[f(\ )_{,\eta}]_{,\xi},\\
\mathscr{L}_{44} &= -\frac{8}{15}(2\pi)^2 v_1(\ ) + \frac{68}{105}\bar{D}[(2\pi\lambda)^2(\ )_{,\eta\eta} + v_1(\ )_{,\xi\xi} - (2\pi)^2\bar{\rho}(\ )_{,tt}],\\
\mathscr{L}_{45} &= \mathscr{L}_{54} = \frac{68}{105}(2\pi\lambda)\bar{D}v_2(\ )_{,\eta\xi},\\
\mathscr{L}_{52} &= \frac{4}{5}\bar{D}((2\pi\lambda)^2 v_1 f(\ )_{,\eta\eta} + [f(\ )]_{,\xi\xi} - (2\pi)^2\bar{\rho}f(\ )_{,tt}),\\
\mathscr{L}_{55} &= -\frac{8}{15}(2\pi)^2 v_1(\ ) + \frac{68}{105}\bar{D}[(2\pi\lambda)^2 v_1(\ )_{,\eta\eta} + (\ )_{,\xi\xi} - (2\pi)^2\bar{\rho}(\ )_{,tt}].
\end{aligned} \tag{A2}$$

APPENDIX B

In the case of antisymmetric displacements, the submatrices $\mathbf{T}_{jk}$ (j, k = 1, 3, 4) appearing in eqns (27) are $N \times N$ square matrices, $\mathbf{T}_{rs}$ (r, s = 2, 5) are $(N+1)\times(N+1)$ square matrices and $\mathbf{T}_{2l}$, $\mathbf{T}_{5l}$, T_{l2}^{T} and T_{l5}^{T} (l = 1, 3, 4) and $N\times(N+1)$ matrices. For symmetric displacements, $\mathbf{T}_{jk}$ (j, k = 1, 3, 4) are $(N+1)\times(N+1)$ square matrices, $\mathbf{T}_{rs}$ (r, s = 2, 5) are $N\times N$ square matrices and $\mathbf{T}_{2l}$, $\mathbf{T}_{5l}$, $\mathbf{T}_{l2}^{T}$ and $\mathbf{T}_{l5}^{T}$ (l = 1, 3, 4) are $(N+1)\times N$ matrices. The dimensions of the submatrices H_{ij} (i, j = 3, 4, 5) can analogously be determined. Finally, **A**, **B**, **C**, **D** and **E** represent proper column matrices which contain the unknown coefficients a_m, b_m, c_m, d_n and e_m, respectively, in the case that the theory proposed in Section 2.1 is used, or the coefficients a_m, b_m, c_m, D_n and E_m, respectively, in the case that the theory developed in Section 2.2 is used.

For the thickness shear deformable analogue of the Love-type theory presented in Section 2.1, the components of the aforementioned Ts and Hs submatrices are given as

$$
\begin{aligned}
&(\mathbf{T}_{11})_{in} = (\lambda_m^2+\nu_1 n^2)\delta_{ni},\\
&(\mathbf{T}_{12})_{in} = (\mathbf{T}_{21})_{in} = -\mu_s\lambda_m n\nu_2\delta_{ni},\\
&(\mathbf{T}_{13})_{in} = (\mathbf{T}_{31})_{in} = -2\lambda_m\nu\Theta_1(n,i),\\
&(\mathbf{T}_{14})_{in} = (T_{41})_{in} = (\mathbf{T}_{15})_{in} = (\mathbf{T}_{51})_{in} = (\mathbf{T}_{24})_{in} = (\mathbf{T}_{42})_{in} = 0,\\
&(\mathbf{T}_{22})_{ni} = (\lambda_m^2\nu_1+n^2)\delta_{ni}+2k_4\Theta_2(n,i),\\
&(\mathbf{T}_{23})_{in} = 2\mu_s n(1+k_4)\Theta_3(n,i)-\frac{1}{\pi}\Theta_4(n,i),\\
&(\mathbf{T}_{25})_{in} = (\mathbf{T}_{52})_{in} = -2k_4\Theta_3(n,i),\\
&(\mathbf{T}_{32})_{in} = 2\mu_s n(1+k_4)\Theta_1(n,i)+\frac{k_4}{\pi}\Theta_4(i,n),\\
&(\mathbf{T}_{33})_{in} = 2\Theta_5(n,i)+(k_5\lambda_m^2+k_4 n^2)\delta_{ni},\\
&(\mathbf{T}_{34})_{in} = (\mathbf{T}_{43})_{in} = k_5\lambda_m\delta_{ni},\\
&(\mathbf{T}_{35})_{in} = (\mathbf{T}_{53})_{in} = -\mu_s k_4 n\delta_{ni},\\
&(\mathbf{T}_{44})_{in} = [k_5+\bar{D}(\lambda_m^2+\nu_1 n^2)]\delta_{ni},\\
&(\mathbf{T}_{45})_{in} = (\mathbf{T}_{54})_{in} = -\mu_s\bar{D}\lambda_m n\nu_2\delta_{ni},\\
&(\mathbf{T}_{55})_{in} = [k_4+\bar{D}(\nu_1\lambda_m^2+n^2)]\delta_{ni},\\
&(\mathbf{H}_{22})_{in} = (H_{33})_{in} = \delta_{ni},\\
&(\mathbf{H}_{23})_{in} = (\mathbf{H}_{25})_{in} = (\mathbf{H}_{32})_{in} = (\mathbf{H}_{34})_{in} = (\mathbf{H}_{43})_{in} = (\mathbf{H}_{35})_{in} = (\mathbf{H}_{52})_{in} = (\mathbf{H}_{53})_{in} = 0,\\
&(\mathbf{H}_{44})_{in} = (\mathbf{H}_{55})_{in} = \bar{D}\delta_{ni}
\end{aligned}
\tag{B1}
$$

where $\lambda_m = \lambda m\pi$, δ_{ni} is Kronecker's symbol and μ_s is 1 or -1 for symmetric or antisymmetric displacements, respectively.

For the shell theory assuming parabolic variation of the thickness shear (Section 2.2), the components of the Ts and Hs submatrices, which are different from the corresponding components given by eqns (B1), are given as

$$
\begin{aligned}
&(\mathbf{T}_{22})_{in} = (\lambda_m^2\nu_1+n^2)[\delta_{ni}+2\bar{D}\Theta_2(n,i)]-\frac{2\mu_s}{\pi}n\bar{D}\Theta_6(i,n)-\frac{1}{2\pi^2}\bar{D}\Theta_{10}(n,i),\\
&(\mathbf{T}_{23})_{in} = 2\mu_s n[1+\bar{D}(\lambda_m^2+n^2)]\Theta_3(n,i)-\frac{1}{\pi}\Theta_4(n,i),\\
&(\mathbf{T}_{24})_{in} = -\mu_s\frac{8}{5}\lambda_m n\nu_2\bar{D}\Theta_3(n,i),\\
&(\mathbf{T}_{25})_{in} = \frac{8}{5}\bar{D}(\lambda_m^2\nu_1+n^2)\Theta_3(n,i),\\
&(\mathbf{T}_{32})_{in} = 2\mu_s n[1+\bar{D}(\lambda_m^2+n^2)]\Theta_1(n,i)+\frac{1}{\pi}\bar{D}(\lambda_m^2+3n^2)\Theta_4(i,n)\\
&\qquad -\mu_s\frac{3}{2\pi^2}\bar{D}n\Theta_8(n,i)-\frac{1}{4\pi^3}\bar{D}\Theta_9(n,i),\\
&(\mathbf{T}_{33})_{in} = 2\Theta_5(n,i)+\bar{D}(\lambda_m^2+n^2)^2\delta_{ni},\\
&(\mathbf{T}_{34})_{in} = (\mathbf{T}_{43})_{in} = -\frac{4}{5}\bar{D}\lambda_m(\lambda_m^2+n^2)\delta_{ni},\\
&(\mathbf{T}_{35})_{in} = (\mathbf{T}_{53})_{in} = \mu_s\frac{4}{5}\bar{D}n(\lambda_m^2+n^2)\delta_{ni},\\
&(\mathbf{T}_{42})_{in} = -\frac{8}{5}\bar{D}\nu_2\lambda_m[\mu_s n\Theta_1(n,i)+\frac{1}{2\pi}\Theta_4(i,n)],\\
&(\mathbf{T}_{44})_{in} = \left[\frac{8}{15}\nu_1+\frac{68}{105}\bar{D}(\lambda_m^2+\nu_1 n^2)\right]\delta_{ni},\\
&(\mathbf{T}_{45})_{in} = (\mathbf{T}_{54})_{in} = -\mu_s\frac{68}{105}\bar{D}\nu_2\lambda_m n\delta_{ni},\\
&(\mathbf{T}_{52})_{in} = \frac{8}{5}\bar{D}\left[(\lambda_m^2\nu_1+n^2)\Theta_3(n,i)-\mu_s\frac{n}{\pi}\Theta_4(n,i)-\frac{1}{4\pi^2}\Theta_7(n,i)\right],
\end{aligned}
$$

$$(\mathbf{T}_{55})_{in} = \left[\frac{8}{15}v_1 + \frac{68}{105}\bar{D}(\lambda_m^2 v_1 + n^2)\right]\delta_{ni},$$

$$(\mathbf{H}_{22})_{in} = \delta_{ni} - 2\bar{D}\Theta_2(n,i),$$

$$(\mathbf{H}_{23})_{in} = -2\mu_s n D\Theta_3(n,i),$$

$$(\mathbf{H}_{25})_{in} = (H_{52})_{in} = \frac{8}{5}\bar{D}\Theta_3(n,i),$$

$$(\mathbf{H}_{32})_{in} = \bar{D}\left[-2\mu_s n\Theta_1(n,i) + \frac{1}{\pi}\Theta_4(i,n)\right],$$

$$(\mathbf{H}_{33})_{in} = [1 + \bar{D}(\lambda_m^2 + n^2)]\delta_{ni},$$

$$(\mathbf{H}_{34})_{in} = (\mathbf{H}_{43})_{in} = \frac{4}{5}\bar{D}\lambda_m\delta_{ni},$$

$$(\mathbf{H}_{35})_{in} = (\mathbf{H}_{53})_{in} = \mu_s\frac{4}{5}\bar{D}n\delta_{ni},$$

$$(\mathbf{H}_{44})_{in} = (\mathbf{H}_{55})_{in} = \frac{68}{105}\bar{D}\delta_{ni}. \tag{B2}$$

The quantities $\Theta_j(n,i)$ ($j = 1, 2, \ldots, 10$) appearing in eqns (B1) and (B2) are given, in integral form, as

$$\Theta_1(n,i) = \alpha\int_0^1 f(\xi)\begin{Bmatrix}\sin(2n\pi\xi)\sin(2i\pi\xi)\\ \cos(2n\pi\xi)\cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_2(n,i) = \alpha\int_0^1 f^2(\xi)\begin{Bmatrix}\cos(2n\pi\xi)\cos(2i\pi\xi)\\ \sin(2n\pi\xi)\sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_3(n,i) = \alpha\int_0^1 f(\xi)\begin{Bmatrix}\cos(2n\pi\xi)\cos(2i\pi\xi)\\ \sin(2n\pi\xi)\sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_4(n,i) = \alpha\int_0^1 \frac{\mathrm{d}f}{\mathrm{d}\xi}\begin{Bmatrix}\sin(2n\pi\xi)\cos(2i\pi\xi)\\ \cos(2n\pi\xi)\sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_5(n,i) = \alpha\int_0^1 f^2(\xi)\begin{Bmatrix}\sin(2n\pi\xi)\sin(2i\pi\xi)\\ \cos(2n\pi\xi)\cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_6(n,i) = \alpha\int_0^1 f(\xi)\frac{\mathrm{d}f}{\mathrm{d}\xi}\begin{Bmatrix}\cos(2n\pi\xi)\sin(2i\pi\xi)\\ \sin(2n\pi\xi)\cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_7(n,i) = \alpha\int_0^1 \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}\begin{Bmatrix}\cos(2n\pi\xi)\cos(2i\pi\xi)\\ \sin(2n\pi\xi)\sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_8(n,i) = \alpha\int_0^1 \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}\begin{Bmatrix}\sin(2n\pi\xi)\sin(2i\pi\xi)\\ \cos(2n\pi\xi)\cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_9(n,i) = \alpha\int_0^1 \frac{\mathrm{d}^3 f}{\mathrm{d}\xi^3}\begin{Bmatrix}\cos(2n\pi\xi)\sin(2i\pi\xi)\\ \sin(2n\pi\xi)\cos(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi,$$

$$\Theta_{10}(n,i) = \alpha\int_0^1 f(\xi)\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}\begin{Bmatrix}\cos(2n\pi\xi)\cos(2i\pi\xi)\\ \sin(2n\pi\xi)\sin(2i\pi\xi)\end{Bmatrix}\mathrm{d}\xi, \tag{B3}$$

where the upper and lower functions appearing in the braces account for antisymmetric and symmetric displacements, respectively.

In any case that $f(\xi)$ is a complicated function of ξ, the quantities $\Theta_j(n,i)$ ($j = 1, 2, \ldots, 10$) can be evaluated numerically. In the oval shell case, the simplicity of expression (29) allows analytical evaluation of $\Theta_j(n,i)$; they are given as

$$\Theta_1(n,i) = \frac{1}{2}\delta_{ni} + \varepsilon\begin{Bmatrix}Z_2(2,n,i)\\ Z_1(2,n,i)\end{Bmatrix},$$

$$\Theta_2(n,i) = \frac{1}{2}(1+\varepsilon^2/2)\delta_{ni} + 2\varepsilon\begin{Bmatrix}Z_1(2,n,i)\\ Z_2(2,n,i)\end{Bmatrix} + \frac{1}{2}\varepsilon^2\begin{Bmatrix}Z_1(4,n,i)\\ Z_1(4,n,i)\end{Bmatrix},$$

$$\Theta_3(n,i) = \frac{1}{2}\delta_{ni} + \varepsilon\begin{Bmatrix}Z_1(2,n,i)\\ Z_2(2,n,i)\end{Bmatrix},$$

$$\Theta_4(n,i) = -4\pi\varepsilon\begin{Bmatrix}Z_2(i,n,2)\\ Z_2(n,i,2)\end{Bmatrix},$$

$$\Theta_5(n,i) = \frac{1}{2}(1+\varepsilon^2/2)\delta_{ni}+2\varepsilon\begin{Bmatrix}Z_2(2,n,i)\\ Z_1(2,n,i)\end{Bmatrix}+\frac{1}{2}\varepsilon^2\begin{Bmatrix}Z_2(4,n,i)\\ Z_1(4,n,i)\end{Bmatrix},$$

$$\Theta_6(n,i) = -4\pi\varepsilon\begin{Bmatrix}Z_2(n,i,2)\\ Z_2(i,n,2)\end{Bmatrix}+\frac{1}{2}\varepsilon\begin{Bmatrix}Z_2(n,i,4)\\ Z_2(i,n,4)\end{Bmatrix},$$

$$\Theta_7(n,i) = -16\pi^2\varepsilon\begin{Bmatrix}Z_1(2,n,i)\\ Z_2(2,n,i)\end{Bmatrix},$$

$$\Theta_8(n,i) = -16\pi^2\varepsilon\begin{Bmatrix}Z_2(2,n,i)\\ Z_1(2,n,i)\end{Bmatrix},$$

$$\Theta_9(n,i) = 64\pi^3\varepsilon\begin{Bmatrix}Z_2(n,2,i)\\ Z_2(i,2,n)\end{Bmatrix},$$

$$\Theta_{10}(n,i) = -4(\pi\varepsilon)^2\delta_{ni}-(4\pi)^2\varepsilon\begin{Bmatrix}Z_1(2,n,i)\\ Z_2(2,n,i)\end{Bmatrix}-8(\pi\varepsilon)^2\begin{Bmatrix}Z_1(4,n,i)\\ Z_2(4,n,i)\end{Bmatrix} \qquad \text{(B4)}$$

where the upper and lower functions appearing in the braces account for antisymmetric and symmetric displacements, respectively, and

$$Z_1(l,n,i) = \alpha\int_0^1 \cos(2l\pi\xi)\cos(2n\pi\xi)\cos(2i\pi\xi)\,d\xi = \begin{cases} \frac{1}{2}\delta_{ni} & l=0 \\ l\neq 0: \begin{cases} 1/4 & n+i=l \\ 1/4 & |n-i|=l \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$$Z_2(l,n,i) = \alpha\int_0^1 \cos(2l\pi\xi)\sin(2n\pi\xi)\sin(2i\pi\xi)\,d\xi = \begin{cases} \frac{1}{2}\delta_{ni} & l=0,\ n\neq 0 \\ l\neq 0: \begin{cases} -1/4 & n+i=l \\ 1/4 & |n-i|=l \end{cases} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(B5)}$$

An attentive observation of the functions Z_1 and Z_2 appearing in expressions (B4) shows that they have a non-zero contribution only if both n and i are even or odd integers. As a result, even and odd displacements (n even or odd integer, respectively) are uncoupled and do not affect each other. Another important remark is that for $\varepsilon = 0$ (circular cylindrical shell), the choice of symmetric or antisymmetric displacements does not affect the final value of the quantities $\Theta_j(n, i)$. Thus, for a circular shell, symmetric and antisymmetric displacements give identical results, provided that $n \neq 0$. For $n = 0$ and antisymmetric displacements the torsional vibration frequencies are obtained. For $n = 0$ and symmetric displacements the axisymmetric vibration frequencies (breathing modes) are obtained.